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hep-th/0001114

# On the classical connection between the WZWN model and topological gauge theories with boundaries

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## Abstract

It is shown, at the level of the classical action, that the Wess-Zumino-Witten-Novikov model is equivalent to a combined BF theory and a Chern-Simons action in the presence of a unique boundary term. This connection relies on the techniques of non-Abelian T-duality in non-linear sigma models. We derive some consistency conditions whose various solutions lead to different dual theories. Particular attention is paid to the cases of the Lie algebras $SO(2,1)$ and $SO(2,1)\times SO(2,1)$. These are shown to yield three dimensional gravity only if the BF term is ignored.

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# 1. Introduction

One of the remarkable features of pure three dimensional gravity is its reformulation as a Chern-Simons theory of gauge connections [1, 2]. The precise relationship between the two theories is well understood when the spacetime has no boundaries. However, since the discovery of the BTZ (Bañados-Teitelboim-Zanelli) black hole solution of three dimensional gravity [3], there has been a new interest in this subject. This is partly in an attempt to provide a statistical mechanical interpretation of the black hole entropy [4, 5, 6]. The main idea behind this approach is, roughly, to treat the horizon of the black hole as a two dimensional boundary of spacetime. This argument is based on some quantum mechanical considerations of black holes. As a consequence, one is forced to deal with a theory of gravity in the presence of boundaries. Imposing then appropriate boundary conditions would lead to physical observables that live on the boundary. After quantization, these extra degrees of freedom (which are absent if the manifold has no boundaries) would correctly account for the black hole entropy.

At this point one might wonder about the precise nature of the two dimensional theory that gives rise to these observables. It has been known for a long time that a Chern-Simons theory, defined on a compact surface, is intimately related to a two dimensional Wess-Zumino-Witten-Novikov (WZWN) model [7]. This connection is in fact made at the quantum level and links the physical Hilbert space of the Chern-Simons theory to the conformal blocks of the WZWN model. If the three dimensional manifold is with boundaries, then the equivalence between the two theories relies on the breaking of the gauge symmetry of the Chern-Simons theory at the boundary [8, 9]. However, in the treatement of three dimensional gravity as a Chern-Simons theory with boundaries, one is forced to preserve gauge invariance as this is indeed what replaces diffeomorphism invariance in gravity [1, 2].

The usual way to proceed in maintaining gauge invariance is as follows: One imposes some boundary conditions on the gauge fields. This requires then the introduction of a corresponding boundary action in order to have a well-defined variational problem. However, the combined action in the bulk and on the boundary is not gauge invariant. A remedy for this problem consists in introducing group-valued fields $g$ living on the boundary. The resulting theory, for very particular boundary conditions, is a Chern-Simons theory *coupled* to a WZWN model [4, 5, 10]. Another point of view in arriving to this conclusion is presented in [11]. The quantisation of the WZWN theory is then believed to account for the entropy of the black hole. It should be emphasised that different boundary conditions yield different conformal field theories on the boundary [12]. A BRST treatment leads also to the same conclusions [13], where different gauge fixings give different theories.

On the other hand, the conventional method in showing the equivalence of Chern-Simons gauge theory and the WZWN model follows a different path [8, 9]. In this procedure one does not introduce (by hand) new dynamical degrees of freedom on the boundary, namely the WZWN fields. In fact, one starts from the Chern-Simons action

$$I_{\mathrm{SC}}\left(A\right) = -\frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{d}^3y\, \epsilon^{\mu\nu\rho}\, \mathrm{tr}\left(A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right) \tag{1}$$

We assume here that $\mathcal{M} = \mathbf{R}\times\mathbf{\Sigma}$, where $\mathbf{\Sigma}$ is a disc whose boundary we denote $\partial\mathbf{\Sigma}$ and whose radial and angular coordinates are $r$ and $\theta$, respectively. The time coordinate $\tau$ parametrises $\mathbf{R}$ and $\epsilon^{\tau r\theta} = 1$. It is clear that gauge invariance, which holds only at the level of the partition function if $\mathcal{M}$ has no boundaries, is broken by the presence of the boundary $\partial\mathcal{M} = \mathbf{R}\times\partial\mathbf{\Sigma}$. The only gauge invariance left is the one for which the gauge parameters reduce to the identity at the boundary. Expanding the action we find

$$I_{\mathrm{SC}}\left(A\right) = -\frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{d}^3y\, \mathrm{tr}\left(2A_\tau F_{r\theta} - A_r\partial_\tau A_\theta + A_\theta\partial_\tau A_r\right) + \frac{k}{4\pi}\int_{\partial\mathcal{M}} \mathrm{d}^2x\, \mathrm{tr}\left(A_\theta A_\tau\right) \quad , \tag{2}$$

where $F_{r\theta} = \partial_r A_\theta - \partial_\theta A_r + [A_r\,,\, A_\theta]$.

If one ignores the boundary action, then $A_\tau$ is a non-dynamical field which forces, in the bulk, the constraint $F_{r\theta} = 0$. The solution to this zero curvature condition is $A_i = L^{-1}\partial_i L$, $(i = r, \theta)$, for some group element $L$. Upon substitution, the Chern-Simons action reduces to a WZWN model. It is not clear though how to justify the fact that the boundary term is not taken into account. The usual given explanation resides in choosing a gauge for which $A_\tau = 0$. However, if one treats $A_\tau$ in the same manner everywhere then one has the constraint $A_\theta = 0$ on the boundary. Solving simultaneously the bulk and the boundary constraints would put conditions on $L$ at the boundary. Furthermore, one might choose to impose, by hand, some other boundary conditions. This in turn introduces a boundary action involving the gauge fields $A_\mu$. Here also different boundary conditions would yield different theories, not necessarily of the WZWN type. In conclusion, the precise connection between Chern-Simons theory, on a manifold with boundaries, and the WZWN model is far from being transparent.

We notice that in all the above mentioned methods, the starting point is the Chern-Simons theory. The WZWN model is obtained either by a direct coupling or by a particular parametrisation of the gauge fields. The aim of this paper is to clarify, at the level of the action and in a classical manner, the nature of the relationship between the WZWN model and Chern-Simons theory. Our starting point is the WZWN model itself. We find that the WZWN theory is dual to a topological BF theory coupled to a Chern-Simons theory in the bulk. The boundary action is unique for this equivalence to hold and no boundary conditions are imposed by hand. This is the same guiding principle for possible boundary terms as that presented in [14]. Our approach makes use of the techniques of non-Abelian T-duality transformations in non-linear sigma models [15]. Here one relies on the existence of isometries and their gauging. In the case of the WZWN model there are two types of isometries. The first consists of the isometries for which the gauge fields live entirely on the boundary and are at most quadratic in the action. The second corresponds to those isometries for which the gauge fields live on the boundary as well as on the bulk. It is this last category of isometries which is used in this study. It has the advantage of allowing for a first order formulation of the WZWN model in terms of gauge fields and Lagrange multipliers where the original fields do not appear anymore. Finally, we specialise to the case of the Lie algebras $SO\,(2,1)$ and $SO\,(2,1) \times SO\,(2,1)$ and explore their relation to three dimensional gravity. We show that, in general, the two theories do not coincide. This is due to the presence of the BF theory action which is usually ignored in the literature when comparing three dimensional gravity to the WZWN model.

# 2. First order formulation of the WZWN model

The action for the WZWN model defined on the group manifold $\mathcal{M}_{\mathcal{G}}$, based on the Lie algebra $\mathcal{G}$, is given by

$$\begin{aligned} S_{\mathrm{WZWN}}\,(g) &= \frac{k}{8\pi}\int_{\partial\mathcal{M}} \mathrm{d}^2x\,\sqrt{|\gamma|}\gamma^{\mu\nu}\,\mathrm{tr}\left(g^{-1}\partial_\mu g\right)\left(g^{-1}\partial_\nu g\right) + \Gamma\,(g) \\ \Gamma\,(g) &= \frac{k}{12\pi}\int_{\mathcal{M}} \mathrm{d}^3y\,\epsilon^{\mu\nu\rho}\,\mathrm{tr}\left(g^{-1}\partial_\mu g\right)\left(g^{-1}\partial_\nu g\right)\left(g^{-1}\partial_\rho g\right) \end{aligned} \tag{3}$$

where $g \in \mathcal{M}_{\mathcal{G}}$ and $\mathcal{M}$ is a three dimensional ball whose boundary is the two dimensional surface $\partial\mathcal{M}$. The metric on this two dimensional worldsheet is denoted by $\gamma_{\mu\nu}$. The remarkable thing about this action is that a variation of the type $g \longrightarrow g + \delta g$ leads to a change in the action, $\delta S_{\mathrm{WZWN}}$, which is an integral over the boundary $\partial\mathcal{M}$ only. The equations of motion are obtained without a need to impose any boundary conditions on the field $g$ or the variation $\delta g$. This property will serve as a guiding principle for our analyses when gauge fields are included. We shall adopt the philosophy of not imposing any boundary conditions on the fields but rather let the equations of motion determine the behaviour of the fields everywhere. This point will be explained in details below.

As it is well-known, the WZWN action has the global symmetry

$$g \longrightarrow LgR \ , \tag{4}$$

where $L$ and $R$ are two constant (more precisely chiral) group elements. Our first step in constructing the dual theory is to gauge this symmetry. We, therefore, introduce two Lie algebra-valued gauge functions $A_\mu$ and $\widetilde{A}_\mu$ transforming as

$$\begin{aligned} A_\mu &\longrightarrow LA_\mu L^{-1} - \partial_\mu L L^{-1} \\ \widetilde{A}_\mu &\longrightarrow R^{-1}\widetilde{A}_\mu R + R^{-1}\partial_\mu R \ . \end{aligned} \tag{5}$$

Since the usual minimal coupling of the gauge fields does not lead to an invariant theory, the gauged WZWN action is found by applying Noether's method [16, 17]. The final result is

$$\begin{aligned} S_{\text{gauge}} &= S_{\text{WZWN}}(g) + I_{\text{SC}}(A) - I_{\text{SC}}\left(\widetilde{A}\right) \\ &+ \frac{k}{4\pi}\int_{\partial\mathcal{M}} \mathrm{d}^2x \,\mathrm{tr}\left[P_+^{\mu\nu}\left(\partial_\mu g g^{-1} A_\nu\right) - P_-^{\mu\nu}\left(g^{-1}\partial_\mu g \widetilde{A}_\nu\right) - P_-^{\mu\nu}\left(A_\mu g \widetilde{A}_\nu g^{-1}\right)\right] \\ &+ \frac{k}{8\pi}\int_{\partial\mathcal{M}} \mathrm{d}^2x \sqrt{|\gamma|}\gamma^{\mu\nu}\,\mathrm{tr}\left(A_\mu A_\nu + \widetilde{A}_\mu \widetilde{A}_\nu\right) \quad . \end{aligned} \tag{6}$$

We have defined, for convenience, the two quantities $P_\pm^{\mu\nu} = \sqrt{|\gamma|}\gamma^{\mu\nu} \pm \epsilon^{\mu\nu}$. Notice also the natural appearance of the Chern-Simons actions corresponding to the gauge fields $A_\mu$ and $\widetilde{A}_\mu$.

There are only two kinds of subgroups of the transformations (4) for which the gauge fields live entirely on the boundary $\partial\mathcal{M}$. This corresponds to the situation when the combination $\left[I_{\text{CS}}(A) - I_{\text{SC}}\left(\widetilde{A}\right)\right]$ vanishes. The first category are the diagonal subgroups where $R = L^{-1}$ and $A_\mu = \widetilde{A}_\mu$ with $L$ and $R$ being Abelian or non-Abelian group elements. The second kind are the axial subgroups for which $R = L$ and $A_\mu = -\widetilde{A}_\mu$, where both $R$ and $L$ are Abelian group elements. The sort of gauging we are interested in corresponds to taking the gauge functions $L$ and $R$ to be two independent, Abelian or non-Abelian, group elements. The Chern-Simons parts of the gauged WZWN action (6) are then present.

The next step towards the dual theory consists in casting the WZWN action in a first order formulation. We begin from the following gauge invariant action

$$S_{\text{total}} = S_{\text{gauge}} - \frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{d}^3y\, \epsilon^{\mu\nu\rho}\,\mathrm{tr}\left(B_\mu F_{\nu\rho}\right) - \frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{d}^3y\, \epsilon^{\mu\nu\rho}\,\mathrm{tr}\left(\widetilde{B}_\mu \widetilde{F}_{\nu\rho}\right) \quad . \tag{7}$$

Here $F_{\mu\nu}$ and $\widetilde{F}_{\mu\nu}$ are the two gauge curvatures correponding, respectively, to $A_\mu$ and $\widetilde{A}_\mu$. The Lie algebra-valued fields $B_\mu$ and $\widetilde{B}_\mu$ are two Lagrange multipliers transforming as $B_\mu \longrightarrow LB_\mu L^{-1}$ and $\widetilde{B}_\mu \longrightarrow R^{-1}\widetilde{B}_\mu R$. The equations of motion of these Lagrange multipliers (or their integration out in a path integral formulation) lead to the constraints $F_{\mu\nu} = \widetilde{F}_{\mu\nu} = 0$. The solutions to these two equations are, up to gauge transformations, given by $A_\mu = h^{-1}\partial_\mu h$

and $\widetilde{A}_\mu = \widetilde{h}^{-1}\partial_\mu \widetilde{h}$ for two group elements $h$ and $\widetilde{h}$. Substituting for $A_\mu$ and $\widetilde{A}_\mu$ in (7) we find that $S_{\text{total}} = S_{\text{WZWN}}\left(hg\widetilde{h}^{-1}\right)$. Therefore, by a change of variables such that $g' = hg\widetilde{h}^{-1}$ (or equivalently by fixing a gauge such that $h = \widetilde{h} = 1$) one recovers the original WZWN model. We conclude that the WZWN model in (3) is equivalent to the theory described by the action in (7).

The gauge invariance of the action $S_{\text{total}}$ allows one to choose a gauge such that $g = 1$. Notice that this gauge choice is not possible for diagonal or axial subgroups. Substituting for $g$, we obtain the sought first order action

$$\begin{aligned} S_{\text{first}} &= I_{\text{SC}}(A) - I_{\text{SC}}\left(\widetilde{A}\right) - \frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{d}^3y\, \epsilon^{\mu\nu\rho}\, \mathrm{tr}\left(B_\mu F_{\nu\rho}\right) - \frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{d}^3y\, \epsilon^{\mu\nu\rho}\, \mathrm{tr}\left(\widetilde{B}_\mu \widetilde{F}_{\nu\rho}\right) \\ &- \frac{k}{4\pi}\int_{\partial\mathcal{M}} \mathrm{d}^2x\, P_-^{\mu\nu}\, \mathrm{tr}\left(A_\mu \widetilde{A}_\nu\right) + \frac{k}{8\pi}\int_{\partial\mathcal{M}} \mathrm{d}^2x\, \sqrt{|\gamma|}\gamma^{\mu\nu}\, \mathrm{tr}\left(A_\mu A_\nu + \widetilde{A}_\mu \widetilde{A}_\nu\right) \quad . \end{aligned} \tag{8}$$

We should mention that the integration over the Lagrange multipliers would always lead to the WZWN model. This follows from the above explanation upon setting $g = 1$, where we obtain $S_{\text{first}} = S_{\text{WZWN}}(g')$ with $g' = h\widetilde{h}^{-1}$. At this point, it is important to emphasise the crucial rôle played by the Lagrange multiplier terms in relating the first order action (8) to the WZWN theory. A further practical manipulation consists in making a field redefinition such that $Q_\mu = B_\mu + \frac{1}{2}A_\mu$ and $\widetilde{Q}_\mu = \widetilde{B}_\mu - \frac{1}{2}\widetilde{A}_\mu$. The first order action takes then the form

$$\begin{aligned} S_{\text{first}} &= -\frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{d}^3y\, \epsilon^{\mu\nu\rho}\, \mathrm{tr}\left[Q_\mu F_{\nu\rho} + \widetilde{Q}_\mu \widetilde{F}_{\nu\rho} - \frac{1}{3}A_\mu A_\nu A_\rho + \frac{1}{3}\widetilde{A}_\mu \widetilde{A}_\nu \widetilde{A}_\rho\right] \\ &- \frac{k}{4\pi}\int_{\partial\mathcal{M}} \mathrm{d}^2x\, P_-^{\mu\nu}\, \mathrm{tr}\left(A_\mu \widetilde{A}_\nu\right) + \frac{k}{8\pi}\int_{\partial\mathcal{M}} \mathrm{d}^2x\, \sqrt{|\gamma|}\gamma^{\mu\nu}\, \mathrm{tr}\left(A_\mu A_\nu + \widetilde{A}_\mu \widetilde{A}_\nu\right) \quad . \end{aligned} \tag{9}$$

In this reformulation the equivalence to the WZWN model is much more transparent and the distinction from pure Chern-Simons theory is evident.

Our first order action (8) has a rich structure in terms of symmetries. Indeed, the Lagrange multiplier terms present in this action correspond to what is commonly known as "$BF$ theories". They are topological theories which have been widely studied (see [18] for a review). A typical characteristic of these theories is their invariance under the finite transformations

$$\begin{aligned} B_\mu &\longrightarrow B_\mu + \mathcal{D}_\mu \alpha = B_\mu + \partial_\mu \alpha + [A_\mu\ ,\ \alpha] \\ \widetilde{B}_\mu &\longrightarrow \widetilde{B}_\mu + \widetilde{\mathcal{D}}_\mu \widetilde{\alpha} = \widetilde{B}_\mu + \partial_\mu \widetilde{\alpha} + \left[\widetilde{A}_\mu\ ,\ \widetilde{\alpha}\right] \end{aligned} \tag{10}$$

for some two arbitrary local functions $\alpha$ and $\widetilde{\alpha}$ evaluated in the Lie algebra $\mathcal{G}$. In our case, this invariance does not hold due to the presence of the boundary. There are, however, two possible ways to restore this invariance. The simplest would be to demande that $\alpha$ and $\widetilde{\alpha}$

vanish when evaluated at the boundary $\partial\mathcal{M}$. The second is to put no restrictions on $\alpha$ and $\widetilde{\alpha}$ and to supply the first order action (8) with the additional boundary term

$$S_{\text{add}} = -\frac{k}{4\pi}\int_{\partial\mathcal{M}} \mathrm{d}^2x\,\epsilon^{\mu\nu}\,\mathrm{tr}\left(\lambda F_{\mu\nu}\right) - \frac{k}{4\pi}\int_{\partial\mathcal{M}} \mathrm{d}^2x\,\epsilon^{\mu\nu}\,\mathrm{tr}\left(\widetilde{\lambda}\widetilde{F}_{\mu\nu}\right) \quad . \tag{11}$$

We associate then to the new fields $\lambda$ and $\widetilde{\lambda}$ the transformations $\lambda \longrightarrow \lambda-\alpha$ and $\widetilde{\lambda} \longrightarrow \widetilde{\lambda}-\widetilde{\alpha}$. The equations of motion corresponding to $\lambda$ and $\widetilde{\lambda}$ force the field strength $F_{\mu\nu}$ and $\widetilde{F}_{\mu\nu}$ to vanish on the boundary $\partial\mathcal{M}$. However, these constraints are already imposed by the Lagrange multipliers $B_\mu$ and $\widetilde{B}_\mu$. In this sense, $\lambda$ and $\widetilde{\lambda}$ are redundant fields and can be set to zero at anytime. In other words, a gauge fixing of the transformations (10) for which $\lambda = \widetilde{\lambda} = 0$ can be made. We will nevertheless keep this additional term in our analyses for later use. Of course we could have also chosen to impose appropriate boundary conditions on the fields themselves. However, this is not in the spirit of our procedure as mentioned above.

The Lagrange multiplier terms in the first order action are seperately invariant under the gauge transformations (5). However, the Chern-Simons parts together with the boundary action are not. It is though possible to recover this invariance by putting restrictions on the gauge parameters as given by

$$L|_{\partial\mathcal{M}} = R|_{\partial\mathcal{M}} = 0 \tag{12}$$

and demande, for quantum invariance, that the topological charges $\Gamma(L)$ and $\Gamma(R)$ are integer-valued.

As a matter of fact, not all of the gauge symmetry (5) is exhausted by the gauge fixing conditon $g = 1$. Indeed, when both $A_\mu$ and $\widetilde{A}$ are present, the first order action is still invariant everywhere under the gauge transformations

$$A_\mu \longrightarrow HA_\mu H^{-1} - \partial_\mu HH^{-1} \quad , \quad \widetilde{A}_\mu \longrightarrow H\widetilde{A}_\mu H^{-1} - \partial_\mu HH^{-1} \tag{13}$$

with no restrictions on the group element $H$ at the boundary. Accordingly, the Lagrange multipliers transform as $B_\mu \longrightarrow HB_\mu H^{-1}$ and $\widetilde{B}_\mu \longrightarrow H\widetilde{B}_\mu H^{-1}$. Since the Lagrange multiplier terms are each gauge invariant under this last symmetry, we have therefore a way of making two Chern-Simons actions (more precisely their difference) gauge invariant. Namely, by coupling them at the boundary in the unique manner as in the first order action (8). This presents an alternative to the usual procedure employed, namely by introducing new dynamical fields, in preserving gauge invariance in Chern-Simons theory.

We return now to our main stream to explore duality. The idea of duality is not to use the equations of motion of the Lagrange multipliers (as these would always lead to the original WZWN theory) but use instead those of the gauge fields. In this context, we distinguish two different situations: If the gauge fields belong to the diagonal or axial subgroups then they appear at most in a quadratic form in the action $S_{\rm first}$. Hence, they can be completely eliminated from the action through their equations of motion (which is equivalent to performing the Gaussian integration over these fields in the path integral). The resulting theory (the dual theory) is another non-linear sigma model but with a different target space metric and a different torsion from those encountred in the original WZWN model. This case has been the subject of many investigations [20] and will not concern us here. The second situation, which will be our main interest, deals with the gauging of subgroups for which the presence of the Chern-Simons action is required. The gauge fields are no longer quadratic in the gauged action. Therefore, they cannot be integrated out from the first order action and the dual theory is certainly not a non-linear sigma model. Let us nevertheless stay at the classical level and examin the equations of motion of the gauge fields $A_\mu$ and $\widetilde{A}_\mu$.

The Lie algebras $\mathcal{G}$ has generators $T_a$ and is specified by the structure constants $f^a_{bc}$ and the trace $\eta_{ab} = {\rm tr}\,(T_a T_b)$, where $\eta_{ab} f^b_{cd} + \eta_{cb} f^b_{ad} = 0$. We do not assume anything on the ivertibility of the invariant bilinear form $\eta_{ab}$. The fields of the first order action are decomposed as $A_\mu = A^a_\mu T_a$, $\widetilde{A}_\mu = \widetilde{A}^a_\mu T_a$, $B_\mu = B^a_\mu T_a$ and $\widetilde{B}_\mu = \widetilde{B}^a_\mu T_a$. We start by calculating the equations of motion in the bulk. The variation of the action (8) with respect to the gauge fields yields

$$\epsilon^{\mu\nu\rho}\,\eta_{ab}\left(F^b_{\nu\rho} + 2\mathcal{D}_\nu B^b_\rho\right) = 0 \quad , \quad \epsilon^{\mu\nu\rho}\,\eta_{ab}\left(\widetilde{F}^b_{\nu\rho} - 2\widetilde{\mathcal{D}}_\nu \widetilde{B}^b_\rho\right) = 0 \ . \tag{14}$$

Acting with the covariant derivatives $\mathcal{D}_\mu$ and $\widetilde{\mathcal{D}}_\mu$ on these equations and using the Bianchi identities, leads to the following consistency relations

$$\epsilon^{\mu\nu\rho}\,\eta_{ab}\,f^b_{cd} F^c_{\mu\nu} B^d_\rho = 0 \quad , \quad \epsilon^{\mu\nu\rho}\,\eta_{ab}\,f^b_{cd} \widetilde{F}^c_{\mu\nu} \widetilde{B}^d_\rho = 0 \ . \tag{15}$$

Among the possible solutions to these consistency equations, two are of particular interest to us. The first is provided by taking $F^a_{\mu\nu} = \widetilde{F}^a_{\mu\nu} = 0$ (we assume for this discussion that $\eta_{ab}$ is invertible). These conditions are as if one is using the equations of motion of the Lagrange multiplier fields $B_\mu$ and $\widetilde{B}_\mu$. Solving these constraints by taking $A_\mu$ and $\widetilde{A}_\mu$ to be pure gauge fields would eventually yield the original WZWN theory. Furthermore, the vanishing of the gauge curvatures (and hence the disappearance of the Lagrange multiplier terms from the action) are the usual equations of motion of pure Chern-Simons theory. This explains why pure Chern-Simons theory (without the BF terms) could be equivalent, in some particular situations, to the WZWN model.

The second solution is given by

$$B_\mu^a = G_{\mu\nu}\,\epsilon^{\nu\alpha\beta}\,F_{\alpha\beta}^a \quad , \quad \widetilde{B}_\mu^a = G_{\mu\nu}\,\epsilon^{\nu\alpha\beta}\,\widetilde{F}_{\alpha\beta}^a \quad , \tag{16}$$

where $G_{\mu\nu}$ is the metric on the three dimensional manifold $\mathcal{M}$. If we substitute these expressions for $B_\mu$ and $\widetilde{B}_\mu$ in the equations of motion (14), we obtain

$$\begin{aligned} \eta_{ab}\left(\epsilon^{\mu\nu\rho}\,F_{\nu\rho}^b + 2\sqrt{|G|}\,G^{\mu\nu}G^{\alpha\beta}\,\mathcal{D}_\alpha F_{\nu\beta}^b\right) &= 0 \\ \eta_{ab}\left(\epsilon^{\mu\nu\rho}\,\widetilde{F}_{\nu\rho}^b - 2\sqrt{|G|}\,G^{\mu\nu}G^{\alpha\beta}\,\widetilde{\mathcal{D}}_\alpha \widetilde{F}_{\nu\beta}^b\right) &= 0 \quad . \end{aligned} \tag{17}$$

These are the equations of motion in the bulk of a three dimensional Yang-Mills theory in the presence of Chern-Simons terms. This observation might explain the origin of the boundary Kac-Moody algebra found in a Yang-Mills-Chern-Simons gauge theory [19]. Therefore, different dual theories to the WZWN model are obtained depending on which solution one chooses for the consistency equations.

In order for the above variational procedure to be well-defined, one needs to specify the boundary conditions of the problem. We choose not to impose by hand boundary conditions on the fields. We rather let the equations of motion play their full rôle and determine for us the boundary conditions. This is achieved by treating the boundary variations in the same manner as those of the bulk. In other words, we demand that $\delta A_\mu$ and $\delta\widetilde{A}_\mu$ are arbitrary both on the bulk and on the boundary. When varying with respect to $A_\mu^a$ and $\widetilde{A}_\mu^a$, one obtains the boundary terms

$$\begin{aligned} &-\frac{k}{4\pi}\int_{\partial\mathcal{M}} \mathrm{d}^2x\,\eta_{ab}\left[P_-^{\rho\mu}\left(\widetilde{A}_\mu^a - A_\mu^a\right) + 2\epsilon^{\rho\mu}\left(B_\mu^a + \mathcal{D}_\mu\lambda^a\right)\right]\delta A_\rho^b \\ &-\frac{k}{4\pi}\int_{\partial\mathcal{M}} \mathrm{d}^2x\,\eta_{ab}\left[P_+^{\rho\mu}\left(A_\mu^a - \widetilde{A}_\mu^a\right) + 2\epsilon^{\rho\mu}\left(\widetilde{B}_\mu^a + \widetilde{\mathcal{D}}\widetilde{\lambda}^a\right)\right]\delta\widetilde{A}_\rho^b \quad , \end{aligned} \tag{18}$$

where we have included the contribution due to the action (11) with $\lambda = \lambda^a T_a$ and $\widetilde{\lambda} = \widetilde{\lambda}^a T_a$. If we choose not to impose the vanishing of $\delta A_\mu^a$ and $\delta\widetilde{A}_\mu^a$ at the boundary, then the equations of motion on the boundary are

$$\begin{aligned} \eta_{ab}\left[P_-^{\rho\mu}\left(\widetilde{A}_\mu^a - A_\mu^a\right) + 2\epsilon^{\rho\mu}\left(B_\mu^a + \mathcal{D}_\mu\lambda^a\right)\right] &= 0 \\ \eta_{ab}\left[P_+^{\rho\mu}\left(A_\mu^a - \widetilde{A}_\mu^a\right) + 2\epsilon^{\rho\mu}\left(\widetilde{B}_\mu^a + \widetilde{\mathcal{D}}_\mu\widetilde{\lambda^a}\right)\right] &= 0 \quad . \end{aligned} \tag{19}$$

This determines for us, in a natural way, the behaviour of the gauge fields at the boundary. Of course, these boundary conditions must be compatible with the bulk equations of motion (14). Since, the boundary action was found in a unique fashion, namely through demanding gauge invariance of the WZWN action, we suspect that the two sets of equations are always compatible. This has been checked at least for the case of the BTZ black hole in the absence of BF terms [21].

# 3. Comparaison with three dimensional gravity

We consider now some special Lie algebras which are relevant to the study of three dimensional gravity. The first of these cases consists in taking the left gauging, $g \longrightarrow Lg$, of the WZWN model. This amounts to setting $\widetilde{A} = 0$ in the first order action (8). We choose a Lie algebra whose generators $T_i$ and $J_i$ satisfy

$$
\begin{aligned}
&[T_i \,,\, T_j] = f_{ij}^k T_k \quad , \quad [T_i \,,\, J_j] = f_{ij}^k J_k \quad , \quad [J_i \,,\, J_j] = 0 \\
&\mathrm{tr}\,(T_i T_j) = 0 \quad , \quad \mathrm{tr}\,(J_i J_j) = 0 \quad , \quad \mathrm{tr}\,(T_i J_j) = \eta_{ij} \quad .
\end{aligned} \tag{20}
$$

We will work with the modified first order action in (9) together with the additional boundary action (11) and expand the different fields there according to

$$
A_\mu = \omega_\mu^i T_i + e_\mu^i J_i \quad , \quad Q_\mu = \theta_\mu^i T_i + v_\mu^i J_i \quad , \quad \lambda_\mu = \chi^i T_i + t^i J_i \quad . \tag{21}
$$

The action for this particular Lie algebra takes then the form

$$
\begin{aligned}
S_{\text{first}}^{(1)} \;=\; & -\frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{d}^3y\, \epsilon^{\mu\nu\rho}\, \eta_{ij}\left[v_\mu^i F_{\nu\rho}^j + 2\theta_\mu^i \mathcal{D}_\nu e_\rho^j - \frac{1}{2} f_{kl}^i \omega_\mu^k \omega_\nu^l e_\rho^j\right] \\
& + \frac{k}{4\pi}\int_{\partial\mathcal{M}} \mathrm{d}^2x\, \eta_{ij}\left[\sqrt{|\gamma|}\gamma^{\mu\nu}\omega_\mu^i e_\nu^j - \epsilon^{\mu\nu}\left(t^i F_{\mu\nu}^j + 2\chi^i \mathcal{D}_\mu e_\nu^j\right)\right] \quad ,
\end{aligned} \tag{22}
$$

where $F_{\mu\nu}^i = \partial_\mu \omega_\nu^i - \partial_\nu \omega_\mu^i + f_{jk}^i \omega_\mu^j \omega_\nu^k$ and $\mathcal{D}_\mu e_\nu^i = \partial_\mu e_\nu^i + f_{jk}^i \omega_\mu^j e_\nu^k$. We remark that the gauge field component $e_\mu^i$ appears linearly in this action. It has, therefore, the function of a Lagrange multiplier which imposes the bulk contraint $\epsilon^{\mu\nu\rho}\,\eta_{ij}\left(\mathcal{D}_\nu \theta_\rho^j - \frac{1}{4} f_{kl}^j \omega_\nu^k \omega_\rho^l\right) = 0$. If a formal solution of the form $\omega_\mu^i = O_\mu^i(\theta)$ exists then the action (22) is effectively a BF theory with just the first term present in the bulk. The true fields are, as expected, the original Lagrange multipliers $\theta_\mu^i$ and $v_\mu^i$. This statement will be of use to us when dealing with three dimensional gravity.

The other example we consider is obtained for left and right gauging, $g \longrightarrow LgR$, of the WZWN model. Here both gauge fields $A_\mu$ and $\widetilde{A}_\mu$ are kept in the first order action (8). We assume that the Lie algebra $\mathcal{G}$ consists of two identical copies (left and right) spanned by the generators $T_i$ and $\widetilde{T}_i$. We have chosen to label the two copies with the same indices. The structure constants are denoted $f_{jk}^i$ for both copies and the trace is such that $\mathrm{tr}\,(T_i T_j) = \mathrm{tr}\left(T_i \widetilde{T}_j\right) = \mathrm{tr}\left(\widetilde{T}_i \widetilde{T}_j\right) = \eta_{ij}$. The two independent gauge fields are decomposed according to

$$
A_\mu = \left(\omega_\mu^i + \alpha\, e_\mu^i\right) T_i \quad , \quad \widetilde{A}_\mu = \left(\omega_\mu^i - \alpha\, e_\mu^i\right) \widetilde{T}_i \quad . \tag{23}
$$

Similarly, the redefined Lagrange multipliers in (9) are written in the form

$$
Q_\mu = \left(\theta_\mu^i + \frac{1}{\alpha}\, v_\mu^i\right) T_i \quad , \quad \widetilde{Q}_\mu = \left(\theta_\mu^i - \frac{1}{\alpha}\, v_\mu^i\right) \widetilde{T}_i \quad . \tag{24}
$$

The action $S_{\rm first}$ for this kind of Lie algebra is given by

$$
\begin{aligned}
S^{(2)}_{\rm first} &= -\frac{k}{2\pi}\int_{\mathcal M} \mathrm{d}^3y\,\epsilon^{\mu\nu\rho}\,\eta_{ij}\left[\theta^i_\mu\left(F^j_{\nu\rho}+\alpha^2 f^j_{kl}e^k_\nu e^l_\rho\right)+2v^i_\mu\,\mathcal{D}_\nu e^j_\rho\right.\\
&- \left.\frac{1}{2}\,\alpha\, f^i_{kl}\left(\omega^k_\mu\omega^l_\nu e^j_\rho+\frac{1}{3}\,\alpha^2 e^k_\mu e^l_\nu e^j_\rho\right)\right]+\frac{k}{2\pi}\int_{\partial\mathcal M}\mathrm{d}^2x\,\eta_{ij}\left\{\alpha^2\sqrt{|\gamma|}\gamma^{\mu\nu}\,\eta_{ij}\,e^i_\mu e^j_\nu\right.\\
&- \left.\epsilon^{\mu\nu}\left[\alpha\,\omega^i_\mu e^j_\nu+\chi^i\left(F^j_{\mu\nu}+\alpha^2\, f^j_{kl}e^k_\mu e^l_\nu\right)+2t^i\,\mathcal{D}_\mu e^j_\nu\right]\right\}\quad , \qquad (25)
\end{aligned}
$$

where we have also expanded the additional fields in (11) as $\lambda=\left(\chi^i+\frac{1}{\alpha}\,t^i\right)T_i$ and $\widetilde{\lambda}=\left(\chi^i-\frac{1}{\alpha}\,t^i\right)\widetilde{T}_i$. Since the above action is cubic in $e^i_\mu$, it is not merely an effective BF theory.

Another reason behind the choice of these particular Lie algebras resides in their close connection with three dimensional gravity. Before entering into the details of the precise relashionship, let us briefly recall the main features of three dimensional gravity in the Palatini formalism (see [22] for a review). The Einstein-Hilbert action in three dimensions,

$$
S^{\rm EH}=\frac{1}{16\pi\kappa}\int_{\mathcal M}\mathrm{d}^3y\sqrt{|G|}\,\left(R-2\Lambda\right)+\int_{\partial\mathcal M}\mathrm{d}^2x\,\mathcal{L}\left(G\right)\quad , \qquad (26)
$$

can be cast in a first order formalism as

$$
S^{\rm EH}_{\rm Palatini}=\frac{1}{16\pi\kappa}\int_{\mathcal M}\epsilon_{IJK}\left[F^{IJ}\left(\Omega\right)\wedge E^K-\frac{\Lambda}{3}\,E^I\wedge E^J\wedge E^K\right]+\int_{\partial\mathcal M}\mathcal{L}\left(\Omega,E\right)\quad . \qquad (27)
$$

The fundamental variables are a one-form connection $\Omega$ and a one-form triad field $E$. The exact nature of the boundary term is not relevant to us here and we refer the reader to [23] for more details. The indices $I,J,K,\ldots=0,1,2$ label an internal space whose flat metric we denote $h_{IJ}$ and $\epsilon_{012}=1$. The spacetime metric is as usual given by $G_{\mu\nu}=h_{IJ}E^I_\mu E^J_\nu$. We take $h_{IJ}$ to be of Lorentzian signature. The curvature two-form is $F^{IJ}\left(\Omega\right)=\mathrm{d}\Omega^{IJ}+\Omega^I{}_K\wedge\Omega^{KJ}$. It is convenient to identify the internal space indices with those of a three dimensional Lie algebra. This is $SO(2,1)$ for Lorentzian gravity and $SO(3)$ for a Euclidean spacetime. The internal metric $h_{IJ}$ is then taken to be proportional to the Killing-Cartan metric of this Lie algebra while $\epsilon^I{}_{JK}=h^{IL}\epsilon_{LJK}$ are its structure constants. Furthermore, for the connection to take value in the Lie algebra, we introduce the new connection (labelled with one index) through the redefinition $\Omega_I{}^J=\epsilon^J{}_{IK}\,\Omega^K$. The one-form connection is such that $\Omega^{IJ}=-\Omega^{JI}$ and is, therefore, metric-preserving (it satisfies the metricity condition).

We return now to our two examples in (22) and (25) and try to find their connection to three dimensional gravity. We specialise to the case when $\eta_{ij}$ and $f^i_{jk}$ describe an $SO(2,1)$ Lie algebra and are proportional to $h_{IJ}$ and $\epsilon^I{}_{JK}$ respectively. We start by examining the first example in (22). As noticed above, the action $S^{(1)}_{\rm first}$ is effectively a BF theory in the bulk and can therefore be identified with $S^{\rm EH}_{\rm Palatini}$ with zero cosmological constant. The boundary Lagrangian $\mathcal{L}\left(\Omega,E\right)$ is unique and corresponds to that of $S^{(1)}_{\rm first}$. The rôle of the triads $E^I_\mu$ is played by the field $v_\mu^i$ and the three dimensional metric is then given by $G_{\mu\nu} = \eta_{ij} v_\mu^i v_\nu^j$. On the other hand, the connection $\Omega_\mu^I$ is related to $O_\mu^i(\theta)$; the solution to the constraints imposed by $e_\mu^i$ on the fields of the action (22). There is, however, a crucial difference between the two actions if one interprets $\omega_\mu^i$ (and not $O_\mu^i(\theta)$) as the spin connection of three dimensional gravity: The variation of (27) with respect to $\Omega_\mu^I$ implies a torsion-free condition whereas a variation of (22) with respect to $\omega_\mu^i$ does not. This is due to the presence of the last two terms in the bulk part of the action (22). In addition, there are more fields in (22) than in (27) and the two theories coincide only if all the Lagrange multipliers in (8) are set to zero. This amounts to setting $\theta_\mu^i = \frac{1}{2}\omega_\mu^i$ and $v_\mu^i = \frac{1}{2}e_\mu^i$ in the action (22).

The situation with the action $S_{\text{first}}^{(2)}$ is much more complex. If the Lagrange multipliers were absent then our action in (25) reduces to $S_{\text{palatini}}^{\text{EH}}$ in the presence of a non vanishing cosmological constant. This can be seen by setting $\theta_\mu^i = \frac{1}{2}\alpha e_\mu^i$ and $v_\mu^i = \frac{1}{2}\alpha\omega_\mu^i$ together with performing an integration by parts in the action (25). In general, however, the two actions are different. This is mainly due to the fact that the field $e_\mu^i$ is cubic in the action (25) and cannot be integrated out. This makes it difficult to determine the fields that play the rôle of the triads and the connection of three dimensional gravity with a non vanishing cosmological constant.

# 4. Conclusions

The WZWN model is a conformal field theory that has been used in the past to reproduce various two dimensional theories like Toda field theories, black holes and others. In this paper we have enlarged this liste by connecting this model to a combination of three dimensional topological BF and Chern-Simons gauge theories defined on a manifold with boundaries. In order for this connection to hold, the boundary action accompanying these topological theories is unique. We arrive to this result by a direct application of non Abelian duality on the WZWN non-linear sigma model. One of our motivations in this work is to provide, at the classical level and at the level of the Lagrangian, a precise relashionship between the WZWN model and three dimensional gravity. This is a special case in our study an two examples are supplied. In the case of the Lie algebra $SO(2,1)$, we find that one obtains three dimensional gravity without a cosmological constant. However, the triads and the connection of gravity are not simply given by the components of the Chern-Simons gauge field. The other example is based on the Lie algebra $SO(2,1) \times SO(2,1)$ and yields three dimensional gravity with a cosmological constant only if the BF theory is not taken into account.

As we have stressed before, the integration over the Lagrange multipliers at any time would yield the original WZWN model. Notice that in both of our examples, the equations of motion of the Lagrange multipliers are simply Einstein's equations (with and without cosmological constant) and the torsion-free condition, if one interprest $e^i_\mu$ and $\omega^i_\mu$, respectively, as the triads and connection of tree dimensional gravity. Therefore, it is *only on-shell* (that is, when Einstein's equations and the torsion-free condition are satisfied) that three-dimensional gravity, with our unique boundary action, is equivalent to the WZWN theory. This statement is, of course, equivalent to ignoring the BF contribution.

One of the interesting problems would be of course to quantise the resulting BF and Cherns-Simons theories taking into account the boundary terms. This might provide a map between the observables of the two dual theories, namely the WZWN model and the topological theory. In the absence of boundaries, a quantum treatment of a combined BF and Chern-Simons has been presented in [24]. There also, the question of which fields might play the rôle of the triads and connection of gravity is raised. We should also mention that we are still at the level the first order action. The next step in the completion of the dualisation procedure is to perform the integration over the gauge field. These appear in a cubic form in the Lagrangian and no known methods allow their complete integration from the path integral. However, a perturbative analyses is always possible. Notice though that there are situations, like the example in (22), where the gauge components appear at most in a quadratic form and without any derivatives acting on them. Integrating them out from the path integral is, in principle, feasible. It might be instructive to start by the quantisation of such models.

**Aknowledgements**: I would like to thank J. Balog and P. Forgács for discussions.

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